# Scrutinizing the Cosmological Constant Problem and a possible resolution

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## Abstract

We suggest a new perspective on the Cosmological Constant Problem by scrutinizing its standard formulation. In classical and quantum mechanics without gravity, there is no definition of the zero point of energy. Furthermore, the Casimir effect only measures how the vacuum energy *changes* as one varies a geometric modulus. This leads us to propose that the physical vacuum energy in a Friedman-Lemaître-Robertson-Walker expanding universe only depends on the time variation of the scale factor $a(t)$. Equivalently, requiring that empty Minkowski space is stable is a principle that fixes the ambiguity in the zero point energy. We describe two different choices of vacuum, one of which is consistent with the current universe consisting only of matter and vacuum energy. The resulting vacuum energy density $\rho_{\rm vac}$ is constant in time and approximately $k_c^2 H_0^2$, where $k_c$ is a momentum cut-off and $H_0$ is the current Hubble constant; for a cut-off close to the Planck scale, values of $\rho_{\rm vac}$ in agreement with astrophysical measurements are obtained. Another choice of vacuum is more relevant to the early universe consisting of only radiation and vacuum energy, and we suggest it as a possible model of inflation.

## I. INTRODUCTION

The Cosmological Constant Problem (CCP) is now regarded as a major crisis of modern theoretical physics. For some reviews of the "old" CCP, see [1–4]. The problem is that simple estimates of the zero point energy, or vacuum energy, of a single bosonic quantum field yield a huge value (the standard calculation is reviewed below). In the past, this led many theorists to suspect that it was zero, perhaps due to a principle such as supersymmetry. The modern version of the crisis is that astrophysical measurements reveal a very small positive value[5][6]:

$$\rho_\Lambda = 0.7 \times 10^{-29}\,\mathrm{g\ cm}^{-3} = 2.8 \times 10^{-47}\ \mathrm{Gev}^4/\hbar^3\mathrm{c}^5. \tag{1}$$

This value is smaller than the naive expectation by a factor of $10^{120}$. This embarrassing discrepancy suggests a conceptual rather than computational error. The main point of this paper is to question whether the CCP as it is currently stated is actually properly formulated. As we will see, our line of reasoning leads to an estimate of the cosmological constant which is much more reasonable, and of the correct order of magnitude.

Let us begin by ignoring gravity and considering only quantum mechanics in Minkowski space. Wheeler and Feynman once estimated that there is enough zero point energy in a teacup to boil all the Earth's oceans. This has led to the fantasy of tapping this energy for useful purposes, however most physicists do not take such proposals very seriously, and in light of the purported seriousness of the CCP, one should wonder why. In fact, there is no principle in quantum mechanics that allows a proper definition of the zero of energy: as in classical mechanics, one can only measure changes in energy, i.e. all energies can be shifted by a constant with no measurable consequences. Similarly, the rules of statistical mechanics tell us that probabilities of configurations are ratios of (conditioned) partition functions, and

these are invariant if the partition functions are multiplied by a common factor as induced by a global shift of the energies. Based on his understanding of quantum electrodynamics and his own treatment of the Casimir effect, Schwinger once said [7], "...the vacuum is not only the state of minimum energy, it is the state of *zero* energy, zero momentum, zero angular momentum, zero charge, zero whatever." A quantum consequence of this for instance is the fact that photons do not scatter off the vacuum energy. All of this strongly suggests that it is impossible to harness vacuum energy in order to do work, which in turn calls into question whether it could be a source of gravitation.

The Casimir effect is often correctly cited as proof of the reality of vacuum energy. However it needs to be emphasized that what is actually measured is the *change* of the vacuum energy as one varies a geometric modulus, i.e. how it depends on this modulus, and this is unaffected by an arbitrary shift of the zero of energy. The classic experiment is to measure the force between two plates as one changes their separation; the modulus in question here is the distance $\ell$ between the plates and the force depends on how the vacuum energy varies with this separation. The Casimir force $F(\ell)$ is minus the derivative of the electrodynamic vacuum energy $E_{\rm vac}(\ell)$ between the two plates, $F(\ell) = -dE_{\rm vac}(\ell)/d\ell$. An arbitrary shift of the vacuum energy by a constant that is independent of $\ell$ does not affect the measurement. For the electromagnetic field, with two polarizations, the well-known result is that the energy density between the plates is $\rho_{\rm vac}^{\rm cas} = -\pi^2/720\ell^4$, which leads to an attractive force.

Let us illustrate the above remark on the Casimir effect with another version of it: the vacuum energy in the finite size geometry of a higher dimensional cylinder. Namely, consider a massless quantum bosonic field on a Euclidean space-time geometry of $S^1 \otimes R^3$ where the circumference of the circle $S^1$ is $\beta$. Viewing the compact direction as spatial, the momenta in that direction are quantized and the vacuum

energy density is

$$\rho_{\rm vac}^{\rm cyl} = \frac{1}{2\beta} \sum_{n\in\mathbb{Z}} \int \frac{d^2\mathbf{k}}{(2\pi)^2} \sqrt{\mathbf{k}^2 + (2\pi n/\beta)^2} = -\beta^{-4}\pi^{3/2}\Gamma(-3/2)\zeta(-3) + \text{const.} \qquad (2)$$

The only difference with the conventional Casimir effect is that for the latter, $2n$ is replaced by $n$ in the above formula. The above integral is divergent, however if one is only interested in its $\beta$-dependence, it can be regularized using the Riemann zeta function giving the above expression. Note that the constant that has been discarded in the regularization is actually at the origin of the CCP. What is measurable is the $\beta$ dependence. One way to convince oneself that this regularization is meaningful is to view the compactified direction as Euclidean time, where now $\beta = 1/T$ is an inverse temperature. The quantity $\rho_{\rm vac}^{\rm cyl}$ is now the free energy density of a single scalar field, and standard quantum statistical mechanics gives the convergent expression:

$$\rho_{\rm vac}^{\rm cyl} = \frac{1}{\beta} \int \frac{d^3\mathbf{k}}{(2\pi)^3} \log\left(1 - e^{-\beta k}\right) = -\beta^{-4} \frac{\zeta(4)}{2\pi^{3/2}\Gamma(3/2)}. \qquad (3)$$

The two above expressions (2, 3) are equal due to a non-trivial functional identity satisfied by the $\zeta$ function: $\xi(\nu) = \xi(1-\nu)$ where $\xi(\nu) = \pi^{-\nu/2}\Gamma(\nu/2)\zeta(\nu)$. (See for instance the appendix in [8].) Either way of viewing the problem allows a shift of $\rho_{\rm vac}^{\rm cyl}$ by an arbitrary constant with no measurable consequences. For instance, such a shift would not affect thermodynamic quantities like the entropy or density since they are derivatives of the free energy; the only thing that is measurable is the $\beta$ dependence.

We now include gravity in the above discussion. Before stating the basic hypotheses of our study, we begin with general motivating remarks. All forms of energy should be considered as possible sources of gravitation, including the vacuum energy. However, if one accepts the above arguments that the zero of energy is not absolutely definable in quantum mechanics, and that only the dependence of the vacuum energy on geometric moduli including the space-time metric is physically measurable,

it then remains unspecified how to incorporate vacuum energy as a source of gravity. One needs an additional principle to fix the ambiguity.

The above observations on the Casimir energy were instrumental toward formulating such a principle, as we now describe. The cosmological Friedmann-Lemaître-Robertson-Walker (FLRW) metric has no modulus corresponding to a finite size analogous to $\beta$, however it does have a time dependent scale factor $a(t)$:

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu = -dt^2 + a(t)^2 d\mathbf{x} \cdot d\mathbf{x}. \tag{4}$$

(We assume the spatial curvature $k = 0$, as shown by recent astrophysical measurements.) When $a(t)$ is constant in time, the FLRW metric is just the Minkowski spacetime metric. This leads us to propose that the dependence of the vacuum energy on the time variation of $a(t)$ is all that is physically meaningful, in analogy with the $\beta$ dependence of $\rho_{\rm vac}^{\rm cyl}$. This idea is stated as a principle below, in terms of the stability of empty Minkowski space, and is at the foundation of our conclusions.

Let us quickly review the standard cosmology. The Einstein equations are

$$G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} \mathcal{R} = 8\pi G T_{\mu\nu} \tag{5}$$

where $G$ is Newton's constant. The stress energy tensor $T_{\mu\nu} = \text{diag}(\rho, p, p, p)$ where $\rho$ is the energy density and $p$ the pressure. The non-zero elements of the Ricci tensor are $R_{00} = -3\,\ddot{a}/a$, $R_{ij} = (2\dot{a}^2 + a\ddot{a})\delta_{ij}$, and the Ricci scalar is $\mathcal{R} = g^{\mu\nu} R_{\mu\nu} = 6\left((\dot{a}/a)^2 + \ddot{a}/a\right)$, where over-dots refer to time-derivatives. The temporal and spatial Einstein equations (5) for the FLRW metric are then the Friedmann equations:

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3} \rho, \tag{6}$$

$$\left(\frac{\dot{a}}{a}\right)^2 + 2\,\frac{\ddot{a}}{a} = -8\pi G\, p. \tag{7}$$

Taking a time derivative of the first equation and using the second, one obtains

$$\dot{\rho} = -3 \left(\frac{\dot{a}}{a}\right) (\rho + p), \tag{8}$$

which expresses the usual energy conservation. The above three equations are thus not functionally independent, the reason being that Bianchi identities relate the two Friedmann equations to the energy conservation equation (8). The total energy density is usually assumed to consist of a mixture of three non-interacting fluids, radiation, matter, and dark energy, $\rho = \rho_{\rm rad} + \rho_m + \rho_\Lambda$, each of which satisfies eq. (8) separately, with $p = w\rho$ for $w = 1/3$, 0 and $-1$ respectively. Then, eq.(8) consistently implies $\dot{\rho}_\Lambda = 0$.

In this paper we will assume that dark energy comes entirely from vacuum energy, $\rho_\Lambda = \rho_{\rm vac}$. The vacuum energy $\rho_{\rm vac}$ is a quantum expectation value,

$$\rho_{\rm vac} = \langle \mathcal{H} \rangle = \langle {\rm vac} | \mathcal{H} | {\rm vac} \rangle, \tag{9}$$

where $\mathcal{H}$ a quantum operator corresponding to the energy density, which is usually associated with $T_{00}$.

Apart from the ambiguity of the zero point energy, several other points should be emphasized. We will be studying the semi-classical Einstein equations, where on the right hand side we include the contribution of vacuum energy $\langle T_{\mu\nu} \rangle = \langle {\rm vac} | T_{\mu\nu} | {\rm vac} \rangle$ for some choice of vacuum state $|{\rm vac}\rangle$. Given the very low energy scale of expansion in the current universe, and the weakness of cosmological gravitational fields, it is very reasonable to assume that there is no need to quantize the gravitational field itself in the present epoch. One hypothesis of the standard formulation of the CCP is that the vacuum stress tensor is proportional to the metric[1]. In an expanding universe, the hamiltonian is effectively time dependent, and there is not necessarily a unique choice of $|{\rm vac}\rangle$, and, in contrast to flat Minkowski space, no Lorentz symmetry argument [15] enforces that $\langle T_{\mu\nu} \rangle \propto g_{\mu\nu}$. One needs extra information that characterizes $|{\rm vac}\rangle$. This implies that $\langle T_{\mu\nu} \rangle$ is not universal since it depends on $|{\rm vac}\rangle$, and thus, for example, cannot always be expressed in terms of purely geometric properties with no reference to the data of $|{\rm vac}\rangle$. One mathematical consistency condition is $D^\mu \langle T_{\mu\nu} \rangle = 0$ if the

various components of the total energy are separately conserved, where $D^\mu$ is the covariant derivative, which is the statement of energy conservation. However this may not follow from $D^\mu T_{\mu\nu} = 0$ since $|\text{vac}\rangle$ may be time dependent. Also, $\langle T_{\mu\nu}\rangle$ is not necessarily expressed in terms of manifestly convariantly conserved tensors such as $G_{\mu\nu}$, again because it depends on $|\text{vac}\rangle$. In fact, the only convariantly conserved geometric tensor that is second order in time derivatives is $G_{\mu\nu}$, and if $\langle T_{\mu\nu}\rangle \propto G_{\mu\nu}$, this would just amount to a renormalization of Newton's constant $G$.

The second point is that if one includes $\langle T_{\mu\nu}\rangle$ as a source in Einstein's equations, then since it depends on $a(t)$ and its time derivatives, doing so can be thought of as studying the back-reaction of this vacuum energy on the geometry. The resulting equations must be solved self-consistently and there is no guarantee that there is a solution consistent with energy-conservation.

Having made these preliminary observations, let us state all of the hypotheses that this work is based upon, which specify either the vacuum states or the nature of their stress-tensor. They are the following:

[i] As a criterion to identify possible vacuum states $|\text{vac}\rangle$, we look for preferred quantization schemes such that $|\text{vac}\rangle$ is an eigenstate of the hamiltonian at all times, which implies there is no particle production.

[ii] We calculate a bare $\rho_{\text{vac},0}$ from the hamiltonian, i.e. $\rho_{\text{vac},0} = \langle\text{vac}|\mathcal{H}|\text{vac}\rangle$ where $\mathcal{H}$ is the quantum hamiltonian energy density operator. The calculation is regularized with a sharp cut-off $k_c$ in momentum space in order to make contact with the usual statement of the CCP.

[iii] We propose the following principle which prescribes how to define a physical $\rho_{\text{vac}}$ from $\rho_{vac,0}$: *Minkowski space that is empty of matter and radiation should be stable, that is, static.* This requires that the physical $\rho_{\text{vac}}$ equal zero when $a(t)$ is constant in time. This leads to a $\rho_{\text{vac}}$ that depends on $a(t)$ and its derivatives, and also the cut-off[9].

[iv] Given this $\rho_{\rm vac}$, we assume the components of the vacuum stress energy tensor have the form of a cosmological constant:

$$\langle T_{\mu\nu}\rangle = -\rho_{\rm vac}\, g_{\mu\nu}. \tag{10}$$

We provide some support for this hypothesis in section III, where we compare our calculation with manifestly covariant calculations performed in the past[19]. We are going to check the consistency of this assumption in the next point [v].

[v] We include $\langle T_{\mu\nu}\rangle$ in Einstein's equations and solve them self-consistently, assuming that vacuum energy and other forms of energy are separately conserved. In other words we study the consistency of the back-reaction of the vacuum energy on the geometry. The consistency condition is $\dot{\rho}_{\rm vac} = 0$, which is equivalent to $D^\mu\langle T_{\mu\nu}\rangle = 0$. There is no guarantee there is such a solution since $\rho_{\rm vac}$ depends on $a(t)$ and its derivatives.

Certainly one may question the validity of these assumptions. However in our opinion, they are rather conservative in that they do not invoke symmetries, particles, nor other, perhaps higher dimensional structures, that are not yet known to exist. The purpose of this paper is to work out the logical consequences of these modest hypotheses. Our main findings are the following:

- If there is a cut-off in momentum space $k_c$, then by dimensional analysis the vacuum energy density has symbolically the "adiabatic" expansion (up to constants):

$$\rho_{vac,0} = k_c^4 + k_c^2\,\widehat{R} + \widehat{R}^2 + \cdots \tag{11}$$

where $\widehat{R}$ is related to the curvature and is a linear combination of $(\dot{a}/a)^2$ and $\ddot{a}/a$, depending on the choice of $|{\rm vac}\rangle$. The principle of the stability of empty Minkowski space [iii] leads us to discard the $k_c^4$ term, but not the other terms since they depend on time derivatives of $a(t)$. Discarding this $k_c^4$ term is just a shift of the zero point energy by a constant, so principle [iii] fixes this ambiguity. Other regularization

schemes, based for example on point-splitting[19], insist on a finite $\rho_{\rm vac}$ and thus discard the first two terms. According to our principles, the second term must be kept since it depends dynamically on the geometry. In the current universe $\widehat{R}$ it is approximately on the order of $H_0^2$, where $H_0$ is the Hubble constant, and if the cut-off $k_c$ is on the order of the Planck energy, then the resulting value of $\rho_{\rm vac}$ is the right order of magnitude in comparison with the measured value (1), namely $\rho_{\rm vac} \approx (k_p H_0)^2 = 3.2 \times 10^{-46}\,{\rm Gev}^4$, using for $H_0$ the present value of the Hubble constant. The $\widehat{R}^2$ is much too small to explain the measured value. We emphasize that our $\rho_{\rm vac}$ is not simply proportional to $H^2$, see eqns. (23,34) below, and is in fact constant in time for the self-consistent solutions that we find. There is nothing special about $H_0$ here, since $\rho_{\rm vac}$ is constant in time; we are simply evaluating it at the present time which involves $H_0$. A practical point of view is that astrophysical observations are telling us that the $k_c^4$ term should be shifted away. More importantly, it remains to determine whether the term that we do keep, $k_c^2 \widehat{R}$, has physical consequences in agreement with observations, which is the main purpose of our study.

The analogy with the Casimir effect is clear both mathematically (compare eqn. (2) and eqn. (20) below), and physically. In the Casimir effect, as one pulls apart the plates in a controlled manner in an experiment, this induces a measurable force. In cosmology the analog of the growing separation of the plates is the expansion itself, which induces an acceleration; the complication is that the effect of this back-reaction must be solved self-consistently, as we will do. There is no measurable effect if the plates are static, which is the analog of the stability of empty Minkowski space, our hypothesis [iii].

- For a universe consisting of only matter and vacuum energy, such as the present universe, there is a choice of $|{\rm vac}\rangle$ with the above $\rho_{\rm vac}$ that leads to a consistent solution if a specific relation between $k_c$ and the Newton constant $G$ is satisfied. By "consistent", we mean $\dot{\rho}_{\rm vac} = 0$. Our solution for $a(t)$ is consistent with present day

astrophysical observations if one ignores the very small radiation component. To our knowledge this choice for $|\text{vac}\rangle$ has not been considered before. Below, we also remark on the cosmic coincidence problem in light of our result.

- For a universe consisting of only radiation and vacuum energy, there is another choice of vacuum, $|\widehat{\text{vac}}\rangle$, that also has a consistent solution, again only for a certain relation between $k_c$ and $G$. This vacuum has been studied before and is referred to as the conformal vacuum in the literature. We suggest that this solution possibly describes inflation, without invoking an inflaton field, and speculate on a scenario to resolve the "graceful exit" problem. We also argue that when $H = \dot{a}/a$ is large, the first Friedmann equation sets the scale $H \sim k_c$, which is the right order of magnitude if $k_c$ is the Planck scale.

The next two sections simply describe these two choices of vacua and analyze the self-consistency of the back-reaction. Our analysis is done using an adiabatic expansion. In the Conclusion, we further discuss our results.

## II. VACUUM ENERGY PLUS MATTER

### A. Choice of vacuum and its energy density

We first review the standard version of the Cosmological Constant Problem. Since a free quantum field is an infinite collection of harmonic oscillators for each wave-vector $\mathbf{k}$, we first review simple quantum mechanical versions in order to point out the difference between bosons and fermions. Canonical quantization of a bosonic mode [16] of frequency $\omega$ yields to a pair of creation and annihilation operators, $a,\ a^\dagger$, with $[a, a^\dagger] = 1$, and a hamiltonian $H = \frac{\omega}{2}(aa^\dagger + a^\dagger a) = \omega(a^\dagger a + \frac{1}{2})$. The boson zero-point energy is thus identified as $\omega/2$. For fermions, the zero point energy has the opposite sign. Fermionic canonical quantization [17] yields to grassmanian operators $b,\ b^\dagger$,

with $\{b, b^\dagger\} = 1, b^2 = {b^\dagger}^2 = 0$, and a hamiltonian $H = \frac{\omega}{2}\left(b^\dagger b - b b^\dagger\right) = \omega(b^\dagger b - \frac{1}{2})$. The fermion zero-point energy is $-\omega/2$.

In a free relativistic quantum field theory with particles of mass $m$ in 3 spatial dimensions, the above applies with $\omega_{\mathbf{k}} = \sqrt{\mathbf{k}^2 + m^2}$, where $\mathbf{k}$ is a 3-dimensional wave-vector. Thus the zero-point vacuum energy density is

$$\rho_{\text{vac}} = \frac{N_b - N_f}{2} \int \frac{d^3\mathbf{k}}{(2\pi)^3} \sqrt{\mathbf{k}^2 + m^2}, \tag{12}$$

where $N_{b,f}$ is the number of bosonic, fermionic particle species. Regularizing the integral with an ultra-violet cut-off $k_c$ much larger than $m$, one finds $\rho_{\text{vac}} \approx (N_b - N_f)k_c^4/16\pi^2$. If $k_c$ is taken to be the Planck energy $k_p$, then $k_c^4/16\pi^2 = 10^{75}$ Gev$^4$. The modern version of the Cosmological Constant problem is the fact that this is too large by a factor of $10^{122}$ in comparison with the measured value. One should also note that in the above calculation, a positive value for $\rho_{\text{vac}}$ requires more bosons than fermions, contrary to the currently known particle content of the Standard Model.

As explained in the Introduction, we are interested in the vacuum energy of a free quantum field in the non-static FLRW background spacetime geometry. For simplicity we consider a single scalar field, with action [22]

$$S = \int dt\, d^3\mathbf{x} \sqrt{|g|} \left( -\frac{1}{2} g^{\mu\nu} \partial_\mu \Phi \partial_\nu \Phi - \frac{m^2}{2} \Phi^2 \right). \tag{13}$$

In order to simplify the explicit time dependence of the action, and thereby simplify the quantization procedure, define a new field $\chi$ as $\Phi = \chi/a^{3/2}$. Then the action (13), after an integration by parts, becomes:

$$S = \int dt\, d^3\mathbf{x} \frac{1}{2} \left( \partial_t \chi \partial_t \chi - \frac{1}{a^2} \vec{\nabla}\chi \cdot \vec{\nabla}\chi - m^2 \chi^2 + \mathcal{A}(t) \chi^2 \right), \tag{14}$$

where

$$\mathcal{A} \equiv \frac{3}{4} \left( \left( \frac{\dot{a}}{a} \right)^2 + 2 \frac{\ddot{a}}{a} \right). \tag{15}$$

The advantage of quantizing $\chi$ rather than $\Phi$ is that all the time dependence is now in $\mathcal{A}$, so that there is no spurious time dependence in the canonical momenta, etc. The field can be expanded in modes:

$$\chi = \int \frac{d^3\mathbf{k}}{(2\pi)^{3/2}} \left( a_{\mathbf{k}}\, u_{\mathbf{k}}\, e^{i\mathbf{k}\cdot\mathbf{x}} + a^\dagger_{\mathbf{k}} u^*_{\mathbf{k}}\, e^{-i\mathbf{k}\cdot\mathbf{x}} \right), \tag{16}$$

where the $a_{\mathbf{k}}$'s satisfy canonical commutation relations $[a_{\mathbf{k}}, a^\dagger_{\mathbf{k}'}] = \delta(\mathbf{k} - \mathbf{k}')$. The function $u_{\mathbf{k}}$ is time dependent and required to satisfy

$$(\partial_t^2 + \omega_{\mathbf{k}}^2) u_{\mathbf{k}} = 0, \qquad \omega_{\mathbf{k}}^2 \equiv (\mathbf{k}/a)^2 + m^2 - \mathcal{A}. \tag{17}$$

The solution is the formal expression

$$u_{\mathbf{k}} = \frac{1}{\sqrt{2W}} \exp\left( i \int^t W(s) ds \right), \tag{18}$$

where $W$ satisfies the differential equation: $W^2 = \omega_{\mathbf{k}}^2 + \frac{3}{4}(\dot{W}/W)^2 - \frac{1}{2}\ddot{W}/W$.

Let us assume that the time dependence is slowly varying, i.e. we make an adiabatic expansion. The above equation can be solved iteratively, where to lowest approximation, $W$ is the above expression with $W$ replaced by $\omega_{\mathbf{k}}$ on the right hand side of the differential equation. However since $\omega_{\mathbf{k}}$ is already second order in derivatives, the additional terms in the above equation are of higher order in the adiabatic expansion. Thus, to lowest order we simply take $W = \omega_{\mathbf{k}}$, and to this order $\dot{u}_{\mathbf{k}} = i\, \omega_{\mathbf{k}} u_{\mathbf{k}}$. With this choice of $u_{\mathbf{k}}$, and to lowest order in the adiabatic expansion, the hamiltonian takes the standard form:

$$H = \frac{1}{2} \int d^3\mathbf{x} \left( \dot{\chi}^2 + \frac{1}{a^2} (\vec{\nabla}\chi)^2 + (m^2 - \mathcal{A})\chi^2 \right) = \frac{1}{2} \int d^3\mathbf{k}\, \omega_{\mathbf{k}} \left( a^\dagger_{\mathbf{k}} a_{\mathbf{k}} + a_{\mathbf{k}} a^\dagger_{\mathbf{k}} \right). \tag{19}$$

Importantly, there are no $a^\dagger_{\mathbf{k}} a^\dagger_{-\mathbf{k}}$ terms, which implies the vacuum $|\text{vac}\rangle$ defined by $a_{\mathbf{k}}|\text{vac}\rangle = 0$ is an eigenstate of $H$ for all times, i.e. there is no particle production, again to lowest order in the adiabatic expansion. By the translational invariance of

the vacuum, for the bare vacuum energy we finally have:

$$\rho_{\text{vac},0} = \frac{1}{V}\langle H \rangle = \int \frac{d^3\mathbf{k}}{(2\pi)^3}\,\sqrt{\mathbf{k}^2 + m^2 - \mathcal{A}}, \tag{20}$$

where $V$ is the volume and we have used $\delta_{\mathbf{k}}(0) = V/(2\pi)^3$. In obtaining the above expression we have properly scaled by redshift factors: $V \to a^3 V$, the cut-off was scaled to $k_c/a$, and we made the change of variables $\mathbf{k} \to a\mathbf{k}$. Comparing the above equation with eq. (2), the analogy with the Casimir effect is clear.

Introducing an ultra-violet cut-off $k_c$ as before, one finds

$$\rho_{\text{vac},0} = \frac{k_c^4}{16\pi^2}\left[1 + \alpha + \frac{\alpha^2}{8}\left(1 + 2\log(\alpha/16)\right)\right], \tag{21}$$

where $\alpha = (m^2 - \mathcal{A})/k_c^2$. Assuming that masses $m$ are all much smaller than the cut-off, we approximate the above expression as:

$$\rho_{\text{vac},0} \approx \frac{k_c^4}{16\pi^2}\left[1 + \frac{m^2}{k_c^2} - \frac{\mathcal{A}}{k_c^2} + \frac{\mathcal{A}^2}{8k_c^4}\right], \tag{22}$$

where we have neglected the logarithmic contribution. It should also be noticed that the $\mathcal{A}^2$ term is beyond the lowest order in the adiabatic expansion.

Now we apply the principle [iii] of the Introduction. In empty Minkowski space, by definition $\dot{a} = \ddot{a} = 0$ and $\rho_{\text{vac}}$ must be zero otherwise empty Minkowski spacetime would not be static due to gravity. Thus, $\rho_{\text{vac},0}$ must be regularized to a physical $\rho_{\text{vac}}$ by subtracting the first two constant terms in brackets:

$$\rho_{\text{vac}} \approx \Delta N\left[\frac{k_c^2}{16\pi^2}\mathcal{A} - \frac{1}{128\pi^2}\mathcal{A}^2\right]. \tag{23}$$

such that $\rho_{\text{vac}} = 0$ when $\dot{a} = \ddot{a} = 0$. Above, we have included multiple species $\Delta N = N_f - N_b$ where $N_{f,b}$ are the numbers of species of fermions,bosons. It is important to observe that in the cylindrical version of the Casimir effect, eq. (2), the analog of the first term above is proportional to $\zeta(-2)k_c/\beta^3 = 0$, so that there is no analog of it in the Casimir effect.

Before proceeding, let us first check that the above expression gives reasonable values. In the present universe, $\dot{a}/a = H_0 = 1.5 \times 10^{-42}$ Gev is the Hubble constant, and $(\dot{a}/a)^2 \approx \ddot{a}/a$. If $k_c$ is taken to be the Planck energy $k_p$, then $\rho_{\rm vac} \sim (k_p H_0)^2 = 3.2 \times 10^{-46}\,{\rm Gev}^4$, which at least is in the ballpark. Fortunately there are more fermions than bosons in the Standard Model of particle physics so that the above expression is positive. Each quark/anti-quark has two spin states, and comes in 3 chromodynamic colors. The electron/positron has two spin states, whereas a neutrino has one. For 3 flavor generations, this gives $N_f = 90$. Each massless gauge boson has two polarizations, 8 for QCD, and 4 for the electro-weak theory, which leads to $N_f - N_b = 62$ including the 4 Higgs fields before spontaneous electro-weak symmetry breaking. The measured value of the vacuum energy can be accounted for with a cut-off about an order of magnitude below the Planck energy [18], $k_c \approx 3 \times 10^{18}$ Gev. We have ignored interactions which modify the value of $\rho_{\rm vac}$, however we expect that they do not drastically change our results. One should also bear in mind that the sharp cutoff $k_c$ is meant to represent a cross-over from the effective theory valid at energy scale well below $k_c$ to that (including gravity) valid above $k_c$.

### B. Consistent backreaction

Let us suppose that the only form of vacuum energy is $\rho_{\rm vac}$ of the last section eq. (23), and that $a(t)$ is varying slowly enough in time that the $\mathcal{A}^2$ term can be neglected. Define the dimensionless constant:

$$g = \frac{3\Delta N}{8\pi} G\, k_c^2 \tag{24}$$

such that

$$\rho_{\rm vac} = \frac{g}{6\pi G}\,\mathcal{A}. \tag{25}$$

Including $\rho_{\text{vac}}$ in the total $\rho$, the first Friedmann equation can be written as

$$\left(1 - \frac{g}{3}\right)\left(\frac{\dot{a}}{a}\right)^2 - \frac{2g}{3}\frac{\ddot{a}}{a} = \frac{8\pi G}{3}\left(\rho_m + \rho_{\text{rad}}\right) \tag{26}$$

We emphasize that we have not modified the Friedman equation; the extra terms on the left-hand side come from $\rho_{\text{vac}}$ which were originally on the right-hand side of the first Friedmann equation.

As we now argue, there is only a consistent solution when $g = 1$. First consider the case where there is no radiation nor matter. Then eq. (26) implies $\ddot{a}/a = (3 - g)(\dot{a}/a)^2/2g$. The pressure can then be found from eq. (7)

$$p_{\text{vac}} = -\frac{1}{g}\,\rho_{\text{vac}}. \tag{27}$$

Thus, the equation of state parameter $w = -1/g$ when there is only $\rho_{\text{vac}}$. However energy conservation requires $\dot{\rho}_{\text{vac}} = 0$, which requires $p_{\text{vac}} = -\rho_{\text{vac}}$, i.e. $g = 1$. The solution is $a(t) \propto e^{Ht}$ for an arbitrary constant $H$, and $\rho_{\text{vac}}$ is independent of time, as a cosmological constant must.

What is not immediately obvious is that a consistent solution can also be found when matter is included, again when $g = 1$. At the current time $t_0$, as usual define the critical density $\rho_c = 3H_0^2/8\pi G$ where $H_0$ is the Hubble constant. The matter and radiation densities scale as $\rho_m/\rho_c = \Omega_m/a^3$ and $\rho_{\text{rad}}/\rho_c = \Omega_{\text{rad}}/a^4$, where $\Omega_m, \Omega_{\text{rad}}$ are the current fractions of the critical density at time $t = t_0$ where $a(t_0) = 1$. The first Friedman equation becomes, when $g = 1$,

$$\frac{2}{3H_0^2}\left[\left(\frac{\dot{a}}{a}\right)^2 - \frac{\ddot{a}}{a}\right] = \frac{\Omega_m}{a^3} + \frac{\Omega_{\text{rad}}}{a^4}. \tag{28}$$

When $\Omega_{\text{rad}} = 0$, the general solution, up to a time translation, is

$$a(t) = \left(\frac{\Omega_m}{\mu}\right)^{1/3}\left[\sinh(3\sqrt{\mu}H_0 t/2)\right]^{2/3}. \tag{29}$$

The constant $\mu$ is fixed by $a(t_0) = 1$. One can check that $\rho_{\text{vac}}$ is indeed constant in time:

$$\frac{\rho_{\text{vac}}}{\rho_c} = \mu. \tag{30}$$

which implies that $\mu + \Omega_m = 1$, i.e. $\mu$ is just $\Omega_{\text{vac}}$ However, when $\Omega_{\text{rad}} \neq 0$, $\rho_{\text{vac}}$ is no longer constant in time. This can be proven directly from the Friedmann equations, or if one wishes, numerically.

Thus, there is a choice of vacuum with a back-reaction that is entirely consistent with the current era. At early times, $a(t) \propto t^{2/3}$, i.e. matter dominated, and at later times grows exponentially, $a(t) \propto \exp(\sqrt{\mu} H_0 t)$, i.e. is dominated by vacuum energy. Given $\Omega_m$, then the equation $a(t_0) = 1$ determines $t_H \equiv H_0 t_0$ and thus the age of the universe. Observations indicate $\Omega_m = 0.266$, and eq. (29) gives $t_H = 0.997$. The reason this is so close to the measured value of $t_H = 0.996$ is that radiation is nearly negligible.

The condition $g = 1$ relates Newton's constant $G$ to the cut-off $k_c$. There are a number of possible interpretations of this curious result. Recall the Planck scale $k_p$ is simply the scale one can define from $G$, but it is not a priori a physicallly meaningful energy scale; rather it is just the scale that one *expects* some form of quantization of the gravitational field to become important. Here, the relation $g = 1$ is a specific relation between the cut-off, Newton's constant G, and the number of particle species, and is unrelated to the quantization of gravity itself. One interpretation is that the cut-off $k_c$ is the fundamental scale and that $G$ is not fundamental, but rather is fixed by the cut-off from $g = 1$. An even more radical interpretation is that quantum fluctuations are actually the origin of gravity itself, which would render the goal of quantizing gravity obsolete.

Finally we wish to make some observations on the so-called cosmic coincidence problem. Simply stated, the problem is that at the present time $t_0$ the densities

of matter and vacuum energy are comparable, and since they evolved at different rates, their ratio would apparently have had to be fine-tuned to differ by many orders of magnitude in the very far past. Let $t_m$ be the time beyond which radiation can be neglected. In the solution eq. (29), $t$ should be replaced by $t - t_m$. To a good approximation, $t_0 - t_m \approx t_0$. Imposing then $a(t_0) = 1$ in equation (29) with $\mu + \Omega_m = 1$ determines $H_0 t_0$ as a function of $\Omega_m$. One can show that $2/3 < H_0 t_0 < \infty$. As stated above, with the present data, $\Omega_m = 0.226$, this gives $H_0 t_0 = 0.997$. Thus the "coincidence" that $\rho_m/\rho_{\rm vac} \approx 0.37$ is now linked to the fact that current measurements give $H_0 t_0$ very close to 1. How would these numbers change if they had been measured in the past, say at time $t' = t_0/2$, over 7 billion years ago when the universe was half as old? Using eq. (29) with $\Omega_m = 0.266$, one finds that at the time $t_0/2$ the Hubble constant was $H' = 1.52\, H_0$ and $H't' = 0.76$. As we did, an observer at that time interprets the solution with $a(t') = 1$, and from eq. (29) we can infer the value of $\Omega'_m$; one finds $\Omega'_m = 0.68$ and $\rho'_m/\rho'_{\rm vac} = 2.13$. Thus for the entire duration of the second half of the universe's history, the product $H't'$ only varied by a factor 3/4 and the ratio $\rho'_m/\rho'_{\rm vac}$ by less than a factor of 6. Actually the evolution of this ratio is very slow. For instance, if one would have measured it at time $t_N = t_0/N$, one would have obtained $(\rho_m/\rho_{\rm vac})(t_N) = 0.61\, N^2$ for $N$ large. Thus one has to go very deep into the past to have a huge difference between $\rho_m$ and $\rho_{\rm vac}$. Furthermore, at these very early times the radiation plays a role and our model breaks down, perhaps with $|{\rm vac}\rangle$ being replaced by $|\widehat{\rm vac}\rangle$ of the next section. An alternative way of summarizing what we have added to the discussion of this issue is the following: if one takes the point of view that the scale of vacuum energy $\rho_{\rm vac}$ is not determined by Planck scale physics, but rather by current day physics, as in our model, then there is much less of a need to explain any fine-tuning in the very far past. All one needs is a high energy cut-off, which is within the framework of low-energy quantum field theory as we currently understand it.

## III. VACUUM ENERGY PLUS RADIATION

### A. Vacuum energy in the conformal time vacuum

In this section, we show that another choice of vacuum $|\widehat{\mathrm{vac}}\rangle$ is consistent with a universe consisting only of vacuum energy and radiation. The quantization scheme is based on conformal time $\tau$, defined as $dt = a d\tau$. Like the choice in the last section, this also simplifies the time dependence of the action (13), and is common in the literature. (See for instance [20, 21] and references therein.) Rescaling the field $\Phi = \phi/a$, and integrating by parts, the action becomes

$$S = \int d\tau d^3x \left( \frac{1}{2}\partial_\tau\phi\partial_\tau\phi - \frac{1}{2}\vec{\nabla}\phi \cdot \vec{\nabla}\phi - \frac{m^2a^2}{2}\phi^2 + \frac{\mathcal{R}a^2}{12}\phi^2 \right), \tag{31}$$

with the Ricci scalar $\mathcal{R} = 6a''/a^3$, where primes indicate derivatives with respect to conformal time $\tau$.

The field can be expanded in modes

$$\phi = \int \frac{d^3\mathbf{k}}{(2\pi)^{3/2}} \left( a_{\mathbf{k}}\, v_{\mathbf{k}}\, e^{i\mathbf{k}\cdot\mathbf{x}} + a^\dagger_{\mathbf{k}} v^*_{\mathbf{k}}\, e^{-i\mathbf{k}\cdot\mathbf{x}} \right), \tag{32}$$

where the $a_{\mathbf{k}}$'s satisfy canonical commutation relations as before. The function $v_{\mathbf{k}}$ is now required to satisfy

$$(\partial_\tau^2 + \widehat{\omega}^2_{\mathbf{k}})v_{\mathbf{k}} = 0, \qquad \widehat{\omega}^2_{\mathbf{k}} \equiv \mathbf{k}^2 + m^2a^2 - \mathcal{R}a^2/6. \tag{33}$$

The analysis of the last section applies with $\mathcal{A}$ replaced by $\mathcal{R}/6$, which leads to:

$$\rho_{\widehat{\mathrm{vac}}} \approx \Delta N \left[ \frac{k_c^2}{96\pi^2}\mathcal{R} - \frac{1}{4608\pi^2}\mathcal{R}^2 \right]. \tag{34}$$

It is instructive to compare the above result with the detailed point-splitting calculation performed in [19] for de Sitter space. The regularization utilized there insists on a finite answer and thus discards the $k_c^4$ and $k_c^2$ term:

$$\langle T_{\mu\nu}\rangle_{\mathrm{ren}} = -g_{\mu\nu} \left( \frac{1}{128\pi^2}(\xi - 1/6)^2\mathcal{R}^2 - \frac{1}{138240\pi^2}\mathcal{R}^2 \right) \tag{35}$$

where $\xi$ is an additional coupling to $\mathcal{R}\Phi$ in the original action. In our calculation $\xi = 0$, and one sees that our simple calculation reproduces the first $\mathcal{R}^2$ term. When $\xi = 1/6$ the theory is conformally invariant and the additional term is the conformal anomaly[23], which our simple calculation has missed. This is not surprising, since the anomaly depends on the spin of the field, and not simply of opposite sign for bosons verses fermions. In any case, in our approximation we are dropping the $\mathcal{R}^2$ terms. What this indicates is that the assumption [iv] in the Introduction is essentially correct if one carefully constructs the full stress tensor in a covariant manner, such as by point-splitting.

### B. Consistent backreaction

In this case define the dimensionless constant:

$$\widehat{g} = \frac{\Delta N}{3\pi} G\, k_c^2 \tag{36}$$

such that

$$\rho_{\widehat{\rm vac}} = \frac{\widehat{g}}{32\pi G}\, \mathcal{R}. \tag{37}$$

Including $\rho_{\widehat{\rm vac}}$ in $\rho$, the first Friedmann equation then becomes

$$\left(1 - \frac{\widehat{g}}{2}\right)\left(\frac{\dot{a}}{a}\right)^2 - \frac{\widehat{g}}{2}\frac{\ddot{a}}{a} = \frac{8\pi G}{3}\left(\rho_m + \rho_{\rm rad}\right). \tag{38}$$

Similarly to what was found in the last section, a consistent solution only exists when $\widehat{g} = 1$, but this time with no matter, $\rho_m = 0$. First consider the case where there is no radiation nor matter. Then eq. (38) implies $\ddot{a}/a = (2-\widehat{g})(\dot{a}/a)^2/\widehat{g}$. Using this, the pressure can again be found from eq. (7)

$$p_{\widehat{\rm vac}} = -\frac{(4-\widehat{g})}{3\widehat{g}}\, \rho_{\widehat{\rm vac}}. \tag{39}$$

Consistency requires the equation of state parameter $w = -1$, i.e. $\widehat{g} = 1$. The solution is $a(t) \propto e^{Ht}$ for some constant $H$, and $\rho_{\widehat{\rm vac}}$ is independent of time.

Now, let us include radiation. At a fixed time $t_i$ define $\rho_i = 3H^2/8\pi G$ where $H$ is a constant equal to $\dot{a}/a$ at the time $t_i$. Now we have to solve (when $\widehat{g} = 1$):

$$\frac{1}{2H^2}\left[\left(\frac{\dot{a}}{a}\right)^2 - \frac{\ddot{a}}{a}\right] = \frac{\Omega_{\rm rad}}{a^4} \tag{40}$$

where $\Omega_{\rm rad} = \rho_{\rm rad}/\rho_i$ at the time $t_i$ where $a(t_i) = 1$. The general solution, up to a time shift, is

$$a(t) = \left(\frac{\Omega_{\rm rad}}{\nu}\right)^{1/4} \sqrt{\sinh(2H\sqrt{\nu}\, t)}, \tag{41}$$

where $\nu$ is a free parameter. Surprisingly, again $\rho_{\widehat{\rm vac}}$ is still a constant,

$$\frac{\rho_{\widehat{\rm vac}}}{\rho_i} = \nu. \tag{42}$$

However this is spoiled if there is matter present (see below). At early times, radiation dominates, $a(t) \propto t^{1/2}$, and at later times vacuum energy dominates, $a(t) \propto \exp(\sqrt{\nu} H t)$.

This choice of vacuum and self-consistent back-reaction is perhaps relevant to the very early universe which consists primarily of radiation and no matter. In fact, at the very earliest times, $H$ is presumably set by the Planck time $t_p = 1/E_p$, which is a much larger scale than $H_0$ by many orders of magnitude. In fact, since the only scale in $\rho_{\rm vac}$ is $k_c$, we expect that higher orders in the adiabatic expansion give $H/k_c$ of order one[24]. For $k_c$ near the Planck scale, then $H$ is roughly of the right scale for inflation. When vacuum energy dominates, $a(t)$ then grows exponentially on a time scale consistent with the inflationary scenario[25–27]. Here this is accomplished without invoking an inflaton field. Many models of inflation typically suffer from the "graceful exit problem", i.e. inflation must come to an end in a relatively short period of time. Based on our work, we suggest the following scenario. Initially there

is only radiation and vacuum energy, which consistently leads to inflation. However as matter is produced, perhaps by particle creation from the vacuum energy, the above solution is no longer consistent. Thus, $|\widehat{\text{vac}}\rangle$ is no longer a consistent vacuum, which suggests that $\rho_{\widehat{\text{vac}}}$ should somehow relax to zero. In support of this idea, we numerically solved eq. (40) with an additional matter contribution on the right hand side equal to $\Omega_m/a^3$. As expected $\rho_{\widehat{\text{vac}}}$ is no longer constant, but decreases in time as shown in Figure 1. Of course, we are already aware that eq.(40) with an additional $\Omega_m \neq 0$ is not consistent with eq.(10) since for a time varying $\rho_{\widehat{\text{vac}}}$ the pressure $p_{\widehat{\text{vac}}} \neq -\rho_{\widehat{\text{vac}}}$; however this plot does indeed show that $\rho_{\widehat{\text{vac}}}$ decreases.

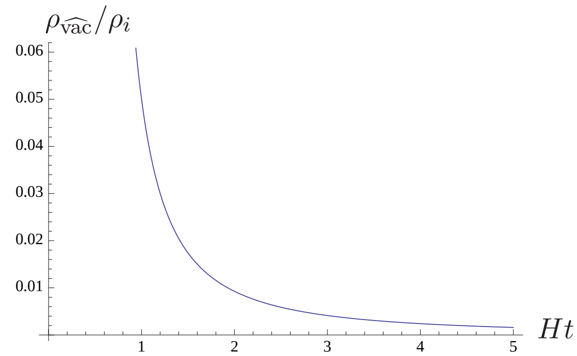


FIG. 1: The vacuum energy $\rho_{\widehat{\text{vac}}}/\rho_i$ as a function of $Ht$ for $\Omega_{\text{rad}} = 0.8$ and $\Omega_m = 0.15$.

## IV. DISCUSSION AND CONCLUSIONS

In this work we have proposed a different point of view on the Cosmological Constant Problem. In analogy with the Casimir effect, we proposed the principle that empty Minkowski space should be gravitationally stable in order to fix the zero point energy which is otherwise arbitrary. In a FLRW cosmological geometry, this leads to a prescription for defining a physical vacuum energy $\rho_{\text{vac}}$ which depends on $\dot{a}$ and $\ddot{a}$. In the current era, this leads to a $\rho_{\text{vac}}$ that is constant in time with

$\rho_{\mathrm{vac}} \approx k_c^2 H_0^2$, which is the correct order of magnitude in comparison to the measured value if the cut-off $k_c$ is on the order of the Planck scale.

We described two different choices of vacua, and studied the self-consistent back-reaction of this vacuum energy on the geometry. One choice of vacuum is consistent with the current matter and dark energy dominated era. Another choice of vacuum is consistent with the early universe consisting of only radiation and vacuum energy, and we suggested that this perhaps describes inflation, and also a resolution to the graceful exit problem. Although our proposals could certainly be further improved, their consequences have at least survived a few checks. The role of higher orders of the adiabatic expansion on the back reaction should be better deciphered.

Both these consistent solutions require a relation between the cut-off $k_c$ and Newton's constant $G$, and we speculated above on possible interpretations of this relation. It remains unclear how to apply the ideas of this work to the time period intermediate between inflation and the current era, where in our scenario, $|\widehat{\mathrm{vac}}\rangle$ would somehow evolve to $|\mathrm{vac}\rangle$, and this is clearly beyond the scope of this paper.

## V. ACKNOWLEDGMENTS

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[6] We will mainly work in energy units. We remind the reader that an inverse length $1/L$ can be converted to an energy $1/L \to \hbar c/L$, and similarly for an inverse time $1/t \to \hbar/t$. The Planck energy $E_p$ and momentum $k_p$ are $E_p = \hbar c k_p = \sqrt{\hbar c^5/G} = 1.2 \times 10^{19}$ Gev. The current value of the Hubble constant is $H_0 = 2.3 \times 10^{-18}\,\mathrm{s}^{-1} = 1.5 \times 10^{-42}$ Gev$/\hbar$. Unless otherwise indicated we set $\hbar = c = 1$.

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[9] After completion of this article we were made aware of some recent works where similar but not identical ideas were pursued[10–13]. In the first 3 of these works, the ADM prescription[14] was invoked to subtract the $k_c^4$ term, which practically speaking is equivalent to our hypothesis [iii]. One difference between these works and ours is that in the former the conformal time vacuum $|\widehat{\mathrm{vac}}\rangle$, described in section III, is utilized; based on our results this vacuum is not consistent with the present universe since it leads to a time-dependent vacuum energy which is ruled out by observations, contrary to our vacuum $|\mathrm{vac}\rangle$ described in section II.

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[15] Of course, we can adopt a hydrodynamic point of view so that the cosmological dark energy fluid is locally characterized by the flat space vacuum energy. But then this vacuum energy is a constant and moduli independent, and thus ill-defined unless we invoke some principle fixing the zero point energy in flat space.

[16] Consider the action for a single bosonic field $\phi$ in zero spatial dimensions: $S = \int dt \left(\frac{1}{2}\partial_t\phi\partial_t\phi - \frac{\omega^2}{2}\phi^2\right)$. The equation of motion is $\partial_t^2\phi = -\omega^2\phi$, thus $\phi$ can be expanded as follows: $\phi = \left(ae^{-i\omega t} + a^\dagger e^{i\omega t}\right)/\sqrt{2\omega}$. Canonical quantization gives $[a, a^\dagger] = 1$, and the hamiltonian is $H = \frac{1}{2}\left(\partial_t\phi\partial_t\phi + \omega^2\phi^2\right) = \frac{\omega}{2}(aa^\dagger + a^\dagger a) = \omega(a^\dagger a + \frac{1}{2})$.

[17] Consider a zero dimensional version of a Majorana fermion with action $S = \int dt \left(i\overline{\psi}\partial_t\overline{\psi} - i\psi\partial_t\psi - \omega\overline{\psi}\psi\right)$ The equation of motion is $\partial_t\psi = -i\omega\overline{\psi}$, and $\partial_t\overline{\psi} = -i\omega\psi$, and thus $\psi, \overline{\psi}$ have the following expansions: $\psi = \frac{1}{\sqrt{2}}\left(be^{i\omega t} + b^\dagger e^{-i\omega t}\right)$, and $\overline{\psi} = \frac{1}{\sqrt{2}}\left(-be^{i\omega t} + b^\dagger e^{-i\omega t}\right)$. Canonical quantization leads to $\{b, b^\dagger\} = 1, b^2 = {b^\dagger}^2 = 0$, and the hamiltonian is $H = \omega\overline{\psi}\psi = \frac{\omega}{2}\left(b^\dagger b - bb^\dagger\right) = \omega(b^\dagger b - \frac{1}{2})$.

[18] If such a scenario has some truth to it, it suggests that not very much "new physics" occurs below the Planck scale.

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[22] Shifting the potential $V(\Phi)$ of a bosonic field does not modify its equation of motion but shifts its stress tensor by a term proportional to $g_{\mu\nu}$. In a hydrodynamic description (see e.g. ref.[28]), this corresponds to a shift of the vacuum energy density $\rho$ by a constant but leaving $(\rho + p)$ unchanged.

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